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B. Sc. (Physics) Part-III, Paper-V, Group-A

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Tensor

Before moving to the details of tensor, we review here some elementary physical laws which will give you some feelings of tensor uses in physics.

$$\vec{a} = \frac{\vec{F}}{m} \quad -(1)$$ → acceleration of a body is proportional to the force acting on it

$$\vec{J} = \sigma \vec{E} \quad -(2)$$ → Electric ~~field~~ current in a medium is proportional to the applied field.

The above physical laws are a special case and apply strictly only to isotropic media (a medium whose properties are same in all directions) or a media which posses high symmetry.

In case of anisotropic media acceleration $\vec{a}$ is not necessarily parallel to the applied force ~~and~~ (Eq. 1) Or the current flows in a direction different from that of the electric field (Eq. 2).

Eq-1 or Eq-2 for anisotropic media can be written in a generalized form. We take Eq 2

$$J_x = \sigma_{xx} E_x + \sigma_{xy} E_y + \sigma_{xz} E_z$$
$$J_y = \sigma_{yx} E_x + \sigma_{yy} E_y + \sigma_{yz} E_z$$
$$J_z = \sigma_{zx} E_x + \sigma_{zy} E_y + \sigma_{zz} E_z$$

J_x, J_y, J_z and E_x, E_y, E_z are respectively the Cartesian components of $\vec{J}$ and $\vec{E}$, and σ_{ij} ($i, j = x, y, z$) are said to be the components of the conductivity tensor

Similar Eq ① can be generalized with $\left(\frac{1}{m}\right)_{ij}$ denoting the components of mass tensor (reciprocal mass tensor) of the particle in the medium.

Tensor Application — mainly in relativistic physics → special theory of relativity, general theory of relativity etc

Conventions & Notations:

Consider an N-dimensional space and let $x^1, x^2, x^3, \ldots, x^N$ be any set of coordinate in this space.

<u>Note</u> that here in writing x^i, i is the superscript on x not the ith power on x.

When it will be needed to write power on x^i we will write it like $(x^i)^2$, $(x^i)^3$, etc

N-dimensional space under the consideration will be denoted by V_n.

A notation $f \equiv f(t)$ will mean that f is a function of t.

Let $\bar{x}^\alpha$ ($1 \leq \alpha \leq N$) be another set of coordinates in the same space V_n. Each of the coordinates x^i will be a function of the N coordinates $\bar{x}^\alpha$ and vice versa

Therefore, we can write

$$x^i = x^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N), \qquad 1 \leq i \leq N \quad \text{——} \; ③$$

$$\bar{x}^\alpha = \bar{x}^\alpha(x^1, x^2, \ldots, x^N), \qquad 1 \leq \alpha \leq N \quad \text{——} \; ④$$

Example: Express the Cartesian and the Spherical polar coordinates as functions of each other.

Soln: We have already discussed curvilinear coordinates in earlier lecture note. We use ~~some~~ those concepts here.

Let x, y, z denote the Cartesian coordinates and r, θ, ϕ the spherical polar coordinates in a three-dimensional space. The coordinates x, y, z are independent of each other. Similarly, r, θ, ϕ are independent of each other.

The coordinates of one set are functions of those of the other set. The Cartesian coordinates are related to the spherical polar coordinates by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

The inverse transformation is given by

$$r = (x^2+y^2+z^2)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\left[\frac{\sqrt{x^2+y^2}}{z}\right], \quad \phi = \tan^{-1}(y/x)$$

You see that (r, θ, ϕ) can be expressed as functions of (x, y, z), and vice versa.

Now differentiating Equations (3) and (4) we can write.

$$dx^i = \sum_{\alpha=1}^{N} \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha, \quad 1 \leq i \leq N \quad (6)$$

$$d\bar{x}^\alpha = \sum_{i=1}^{N} \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i, \quad 1 \leq \alpha \leq N. \quad (7)$$

If we use Einstein's summation convention, it will simplify the above notations.

Einstein's Summation convention: If an index (except N) is repeated in a term, sumation over it from 1 to N in implied.

Therefore, we can write

$$d x^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha, \quad 1 \le i \le N \quad (8)$$

$$d\bar{x}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} d x^i, \quad 1 \le \alpha \le N \quad (9)$$

Note that α, appears twice on the rhs of Eq. (8) and i appears twice on the rhs of Eq. (9), therefore, summation over these from 1 to N is applied in the respective Equations.

We use this convention throughout this whole discussion of tensor analysis.

It is to be noted that if an index appears only once in any term, it has a definite value (any value from 1 to N) This index is called as free index. In Eq. (8), i is free index and in Eq. (9), α is free index. Further, we drop $1 \le i \le N$ in Eq. (8) and $1 \le \alpha \le N$ in Eq. (9). this should be understood.

free index

$$d x^i = \frac{\partial x^{i}}{\partial \bar{x}^{\alpha}} d\bar{x}^{\alpha}$$

dummy index

$$d\bar{x}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} d x^i$$

Dummy index → An index which is repeated and over which summation is implied is called a dummy index. Dummy index can be replaced by any other index which does not appear in the same term.